Ext 2

NORTH SYDNEY BOYS HIGH SCHOOL

2015 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – **3 hours**
- For Section I, shade the correct box on the sheet provided.
- For Section II, write in the booklet provided.
- Each new question is to be started on a **new page.**
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Marks may be deducted for illegible or incomplete working
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Ireland
- O Mr Lin
- O Mr Weiss

Student Number

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/15	/15	/15	/15	/15	/100

Section I: Objective Response

Mark your answers on the multiple choice answer sheet provided by shading the correct box.

1. One solution to the equation $x^4 - 6x^3 + 26x^2 - 46x + 65 = 0$ is $x = 2 - 3i$. 1

   Another solution is:

   (A) $1 - 2i$ (B) $-1 - 2i$ (C) $-2 - i$ (D) $-2 + i$

2. What restrictions must be placed on p if α, β, γ are the three non-zero real roots of the equation $x^3 + px - 1 = 0$? 1

   (A) $p > 0$, p is real (B) $p < 0$, p is real

   (C) $p \geq 0$, p is real (D) $p \leq 0$, p is real

3. Consider the two statements: 1

   I: $$\int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}}$$

   II: $$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$$

   Which of following is true?

   (A) Neither statement (B) Statement I only

   (C) Statement II only (D) Both statements

4. The polynomial equation $x^3 + 4x^2 - 2x - 5 = 0$ has roots α, β, γ. 1

Which of the following equations has roots $\alpha^2, \beta^2, \gamma^2$?

(A) $x^3 - 20x^2 - 44x - 25 = 0$ (B) $x^3 - 20x^2 + 44x - 25 = 0$

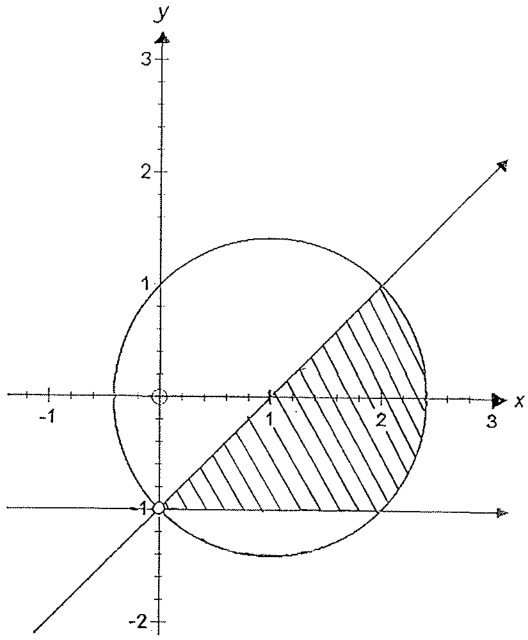
(C) $x^3 - 4x^2 + 5x - 1 = 0$ (D) $x^3 + 4x^2 + 5x - 1 = 0$

5. Which of the following is an expression for $\int \frac{dx}{\sqrt{7-6x-x^2}}$? 1

(A) $\sin^{-1}(\frac{x-3}{2})+c$ (B) $\sin^{-1}(\frac{x+3}{2})+c$

(C) $\sin^{-1}(\frac{x-3}{4})+c$ (D) $\sin^{-1}(\frac{x+3}{4})+c$

6. The shaded area in the Argand diagram below could be described by which pair of inequalities? 1



(A) $|z-1| \le \sqrt{2}$ and $0 \le \arg(z-i) \le \frac{\pi}{4}$ (B) $|z-1| \le \sqrt{2}$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$

(C) $|z-1| \le 1$ and $0 \le \arg(z-i) \le \frac{\pi}{4}$ (D) $|z-1| \le 1$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$

7. The region bounded by the parabola $y = x^2$ and the x-axis between $x = 0$ and $x = 1$ is rotated about the line $x = 2$ to form a solid of volume V.

Which of the following is an expression for V? 1

(A) $\pi \int_0^1 (1-x)^2 \, dy$ (B) $\pi \int_0^1 (1^2 - x^2) \, dy$

(C) $\pi \int_0^1 \left[(2-x)^2 - 1^2\right] dy$ (D) $\pi \int_0^1 \left[2^2 - (2-x)^2\right] dy$

8. Which of the following is equal to $\int \sin^3 x \, dx$? 1

(A) $\frac{1}{4}\sin^4 x + c$

(B) $-\cos x + \frac{1}{3}\cos^3 x + c$

(C) $-\cos x - \frac{1}{3}\cos^3 x + c$

(D) $\cos x - \frac{1}{3}\cos^3 x + c$

9. The equation of the tangent to the ellipse $x = 3\cos\theta, \ y = 2\sin\theta$ at the point where $\theta = \frac{\pi}{3}$ is: 1

(A) $6\sqrt{3}x - 4y - 5\sqrt{3} = 0$

(B) $2x - 3\sqrt{3}y - 12 = 0$

(C) $2x + 3\sqrt{3}y - 12 = 0$

(D) $6\sqrt{3}x + 4y - 5\sqrt{3} = 0$

10. $P(4, 25)$ is a point on the rectangular hyperbola $xy = 100$. 1

The tangent at P cuts the hyperbola's asymptotes at Q and R.

The area of ΔOQR (where O is the origin) is:

(A) $200\sqrt{2} \quad u^2$

(B) $2\sqrt{50} \quad u^2$

(C) $100 \quad u^2$

(D) $200 \quad u^2$

Section II: Short Answer

Question 11 (15 marks) Commence a NEW page

(a) Evaluate $\int_0^{\frac{\pi}{3}} \sec^4 x \tan x \, dx$ 3

(b) Find $\int \frac{dx}{\sqrt{x^2 - 8x + 25}}$ 2

(c)

(i) Resolve $\frac{9 + x - 2x^2}{(1-x)(3+x^2)}$ into partial fractions. 2

(ii) Hence find $\int \frac{9 + x - 2x^2}{(1-x)(3+x^2)} dx$ 2

(d) Use the substitution $t = \tan(\frac{x}{2})$ to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1 - \cos x} dx$ 3

(e)

(i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 1

(ii) Hence evaluate $\int_0^{\pi} x \sin x \, dx$ 2

Question 12 (15 marks) Commence a NEW page **Mark**

(a) Find the square roots of $9-40i$. (Give your answer in the form $a+ib$) 2

(b) Express $z=\sqrt{3}+i$ in modulus-argument form. 1

(c) (i) Find the Cartesian equation of the locus represented by $2|z|=3\,(z+\bar{z})$. 2

(ii) Sketch the locus on an Argand diagram. 1

(d) Given that $z=\cos\theta+i\,\sin\theta$,

(i) Show that $z^n+z^{-n}=2\cos n\theta$ 2

(ii) Hence solve the equation $2z^4-z^3+3z^2-z+2=0$ 3

(e) P is a point in the complex plane representing the complex number z, where

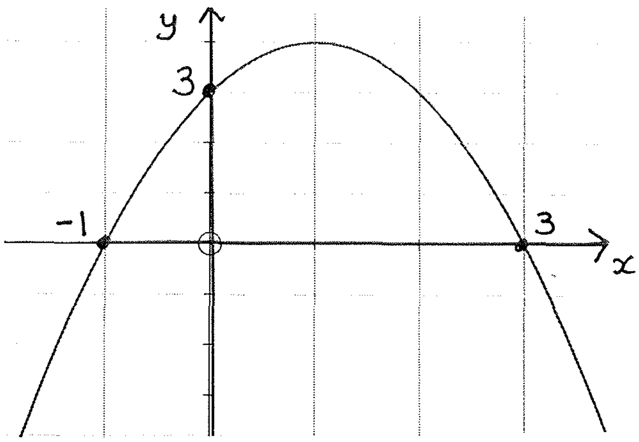
z satisfies $|z-2|=2$ and $0<\arg z<\dfrac{\pi}{2}$.

(i) Sketch the locus described by these conditions. 1

(ii) Find the value of the real number k if $\arg(z-2)=k\arg(z^2-2z)$. 3

Question 13 (15 marks) Commence a NEW page

(a) Let $f(x) = -(x-3)(x+1)$. The graph shown below depicts $y = f(x)$:



On separate diagrams, sketch the following graphs without using calculus.

Indicate any asymptotes, intercepts or other important features.

(i) $y = f(|x|)$ 2

(ii) $y = \dfrac{1}{f(x)}$ 2

(iii) $y = e^{f(x)}$ 2

(iv) $y^2 = f(x)$ 2

(b)

(i) State the domain and range of $y = \cos^{-1}(e^x)$ 2

(ii) Without using calculus, sketch the graph of $y = \cos^{-1}(e^x)$, showing clearly any intercepts and the equations of any asymptotes. 2

(c) For the curve defined by $3x^2 + y^2 - 2xy - 8x + 2 = 0$ find the coordinates of the points on the curve where the tangent is parallel to the line $y = 2x$. 3

Question 14 (15 marks) Commence a NEW page **Mark**

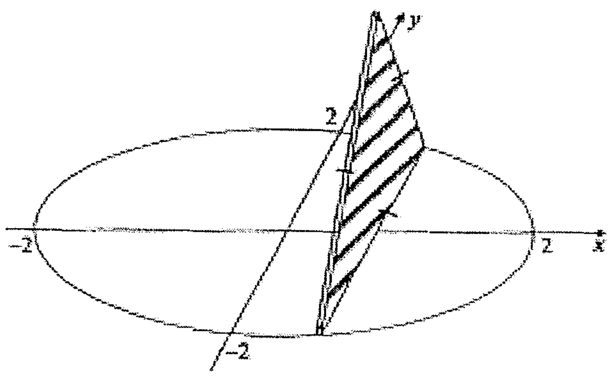
(a)

(i) If α is a root of $P(x)$ with multiplicity n, show that α is also a root of $P'(x)$ with multiplicity $n-1$. 1

(ii) Given $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$ has a triple root, factorise $P(x)$ into its linear factors. 3

(b) Using the method of cylindrical shells, find the volume generated by revolving the area bounded by the lines $x = \pm 2$ and the hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$ about the y-axis. 4

(c) The diagram below shows a cross-sectional slice of a solid whose base is the region enclosed by the circle $x^2 + y^2 = 4$.
Each such cross-section is an equilateral triangle.
Find the volume of the solid. 3



(d) Suppose that $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$ and the equation $P(x) = 0$ has roots $\alpha, \beta, \gamma, \delta$,

(i) Show that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2$ 2

(ii) Hence prove that the equation $P(x) = 0$ has precisely two real roots. 2

Question 15 (15 marks) Commence a NEW page **Mark**

(a) $P(5p, \frac{5}{p})$ and $Q(5q, \frac{5}{q})$, $p, q > 0$, are two variable points on the hyperbola $xy = 25$.

(i) Derive the equation of the chord PQ. 2

(ii) State the equations of the tangents at P and Q. 1

(iii) If the tangents at P and Q intersect at R, find the coordinates of R. 2

(iv) If the secant PQ passes through the point $(15, 0)$, find the locus of R. 2

(b) Points P and Q are the endpoints of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

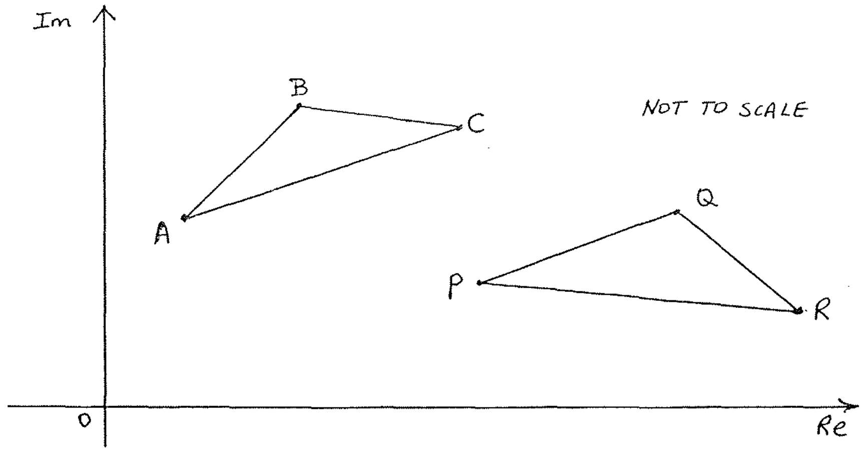
If the parameters at P and Q are θ and ϕ respectively, show that the ellipse's eccentricity is given by $e = \frac{\sin(\theta - \phi)}{\sin\theta - \sin\phi}$. 3

(c) A sequence of numbers T_n, $n = 1, 2, 3 ...$, is defined by $T_1 = 2$, $T_2 = 0$ and $T_n = 2T_{n-1} - 2T_{n-2}$, for $n = 3, 4, 5 ...$ **5**

Use mathematical induction to show that $T_n = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}$, $n = 1, 2, 3 ....$

Question 16 (15 marks) Commence a NEW page **Mark**

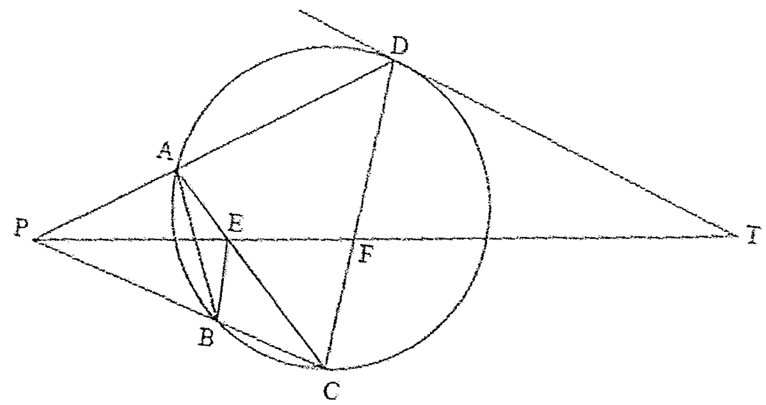
(a)



The points A, B, C, represent the complex numbers z_1, z_2, z_3 respectively.
The points P, Q, R, represent the complex numbers w_1, w_2, w_3.

If $\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{w_2 - w_1}{w_3 - w_1}$ then prove that ΔABC is similar to ΔPQR. **3**

(b) $ABCD$ is a cyclic quadrilateral. DA produced and CB produced meet at P.
T is a point on the tangent at D to the circle through A, B, C and D.
PT cuts CA and CD at E and F respectively. $TF = TD$.



Copy this diagram into your writing booklet.

(i) Show that $AEFD$ is a cyclic quadrilateral. 2

(ii) Show that $PBEA$ is a cyclic quadrilateral. 3

(c) Let $I_n = \int_0^1 (1 - x^2)^n dx$ and $J_n = \int_0^1 x^2(1 - x^2)^n dx$

(i) Apply integration by parts to I_n to show that $I_n = 2n\, J_{n-1}$ **2**

(ii) Hence show that $I_n = \frac{2n}{2n+1} I_{n-1}$ **2**

(iii) Show that $J_n = I_n - I_{n+1}$ and hence deduce that $J_n = \frac{1}{2n+3} I_n$ **2**

(iv) Hence write down a reduction formula for J_n in terms of J_{n-1} **1**

END OF EXAMINATION

2015 Extension 2 Assessment Task 3 Suggested Solutions.

SECTION 1 - OBJECTIVE RESPONSE

1. A
2. B
3. D
4. B
5. D
6. B
7. C
8. B
9. C
10. D

/10

Q.11

(a) $I = \int_0^{\frac{\pi}{3}} \sec^4 x \tan x \, dx$

$= \int_0^{\frac{\pi}{3}} \sec^3 x \cdot \sec x \tan x \, dx$

$= \left[\frac{\sec^4 x}{4} \right]_0^{\frac{\pi}{3}}$ by standard integrals ✓✓ uses table correctly

$= \frac{2^4}{4} - \frac{1^4}{4}$

$= \frac{15}{4}$ ✓ Correct answer

[ALT: $I = \int \sec^2 x \cdot \sec^2 x \tan x \, dx$

$= \int (1 + \tan^2 x) \cdot \tan x \cdot \sec^2 x \, dx$

$= \int (\tan x + \tan^3 x) \sec^2 x \, dx$

$= \left[\frac{\tan^2 x}{2} + \frac{\tan^4 x}{4} \right]_0^{\pi/3}$

$= \frac{3}{2} + \frac{9}{4}$

$= \frac{15}{4}$.]

$\boxed{Q11}$ - ctd

(b) $$I = \int \frac{dx}{\sqrt{x^2 - 8x + 25}}$$

$$= \int \frac{dx}{\sqrt{(x-4)^2 + 3^2}}$$ ✓ completes square

$$= \ln\left(x - 4 + \sqrt{(x-4)^2 + 9}\right) + C$$ ✓ uses table correctly

(by standard integrals)

(c) (i) $$\frac{9 + x - 2x^2}{(1-x)(3+x^2)} \equiv \frac{A}{1-x} + \frac{Bx + C}{3 + x^2}$$

$$\therefore 9 + x - 2x^2 \equiv A(3 + x^2) + (Bx + c)(1-x)$$
$$= (A-B)x^2 + (B-c)x + (3A+c)$$

$$\therefore \begin{array}{llll} & A - B = -2 & (1) \\ & B - c = 1 & (2) \\ & 3A + c = 9 & (3) \\ (1)+(2)\to & A - c = -1 & (4) \\ (3)+(4)\to & 4A = 8 & \therefore A = 2 \end{array}$$

$$\therefore c = 3, \quad B = 4.$$

$$\therefore \frac{9 + x - 2x^2}{(1-x)(3+x^2)} \equiv \frac{2}{1-x} + \frac{4x + 3}{3 + x^2}$$ ✓✓

(ii) $$I = \int \left[\frac{2}{1-x} + \frac{4x}{3+x^2} + \frac{3}{3+x^2}\right] dx$$

$$\therefore I = -2\ln|1-x| + 2\ln(3+x^2) + \sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + C$$ ✓✓

$\boxed{Q11}$ - ctd

(d) $I = \displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\, dx$

Let $t = \tan\left(\frac{x}{2}\right)$

$\therefore \quad dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$

$2\,dt = \left(1+\tan^2\frac{x}{2}\right) dx$

ie $\quad dx = \dfrac{2}{1+t^2}\, dt$

$\begin{cases} x = \frac{\pi}{3} \to t = \frac{1}{\sqrt{3}} \\ x = \frac{\pi}{2} \to t = 1 \end{cases}$

✓ makes dx subst^n & changes limits

Thus $\quad I = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$

$= \displaystyle\int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{t^2}\, dt$

✓ correctly simplifies

$= \left[-\frac{1}{t}\right]_{\frac{1}{\sqrt{3}}}^{1} \quad = -1+\sqrt{3}.$

✓ correct answer

(e) (i) If $I = \displaystyle\int_0^a f(a-x)\, dx$, let $u = a-x$

$\therefore \quad du = -dx$

$x=0 \to u=a, \quad x=a \to u=0$

$\therefore \quad I = \displaystyle\int_a^0 f(u)\cdot -du$

$= \displaystyle\int_0^a f(u)\, du = \int_0^a f(x)\, dx.$

✓ correct proof

(ii) $I = \displaystyle\int_0^\pi x\sin x\, dx = \int_0^\pi (\pi - x)\sin(\pi - x)\, dx$ from (i)

$= \displaystyle\int_0^\pi \pi\sin x\, dx - \int_0^\pi x\sin x\, dx$

✓ uses part (i)

$\therefore \ 2I = \pi\displaystyle\int_0^\pi \sin x\, dx = \pi\left[-\cos x\right]_0^\pi$

$= \pi\left[1-(-1)\right] = 2\pi$

$\therefore \quad I = \pi.$

✓ correct answer

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Q12

(a) Let $z^2 = (a+ib)^2 = 9-40i$

$\therefore \quad a^2 - b^2 = 9$

and $\quad 2abi = -40i \quad \therefore ab = -20.$

By inspection, $a=5, b=-4$ or $a=-5 \quad b=4$

i.e. square roots are $5-4i$ and $-5+4i$.

(b) $z = \sqrt{3} + i = 2\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$z = 2\left(\cos\frac{\pi}{6} + \sin\frac{\pi}{6}i\right)$

(c) (i) Let $z = x+iy$, x,y real

$2|z| = 3(z+\bar{z})$

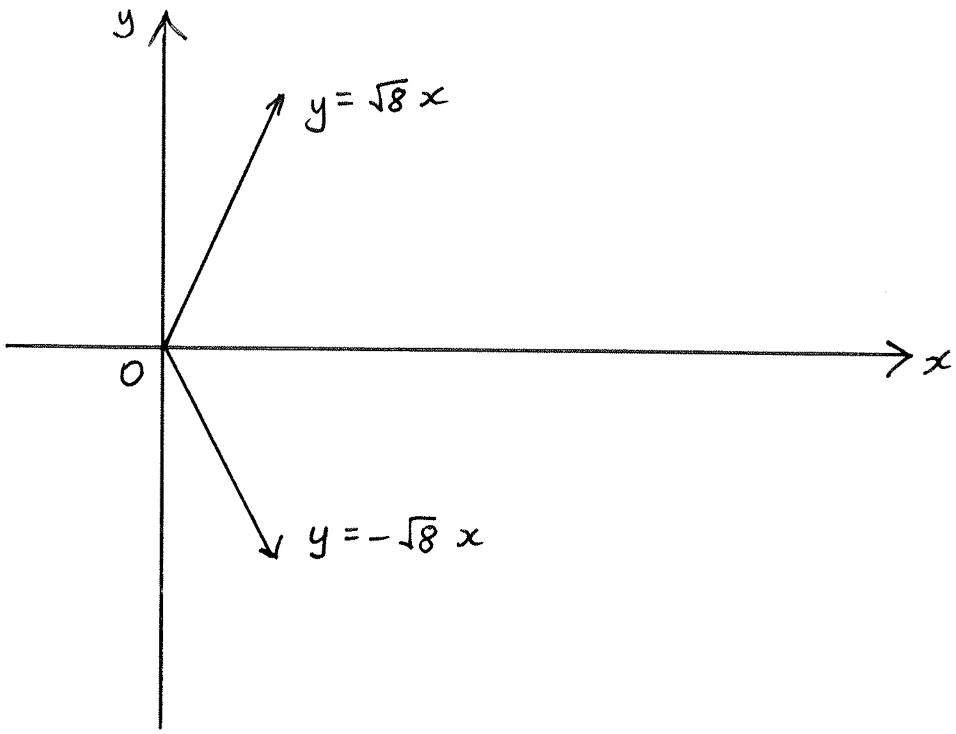
$4|z|^2 = 9(z+\bar{z})^2$ and $x \geq 0$

$4(x^2+y^2) = 9(2x)^2$

$4y^2 = 32x^2$

$y^2 = 8x^2 \quad , \quad x \geq 0$

(ii)



$\boxed{Q12}$ - ctd

(d) (i) $z = \cos\theta + i\sin\theta$

$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

$z^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos n\theta - i\sin n\theta$

$\therefore z^n + z^{-n} = \cos n\theta + \cancel{i\sin n\theta} + \cos n\theta - \cancel{i\sin n\theta}$

$= 2\cos n\theta$.

✓✓

(ii) $2z^4 - z^3 + 3z^2 - z + 2 = 0$

$2z^4 + 2 - z^3 - z + 3z^2 = 0$

$2(z^4+1) - (z^3+z) + 3z^2 = 0$

$(\div z^2 \rightarrow)$ $2(z^2 + z^{-2}) - (z + z^{-1}) + 3 = 0$

$2(2\cos 2\theta) - 2\cos\theta + 3 = 0$

$4\cos 2\theta - 2\cos\theta + 3 = 0$

✓

$4(2\cos^2\theta - 1) - 2\cos\theta + 3 = 0$

$8\cos^2\theta - 2\cos\theta - 1 = 0$

$(2\cos\theta - 1)(4\cos\theta + 1) = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{1}{4}$

✓

if $\cos\theta = \frac{1}{2}$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$

if $\cos\theta = -\frac{1}{4}$, $\sin\theta = \pm\frac{\sqrt{15}}{4}$

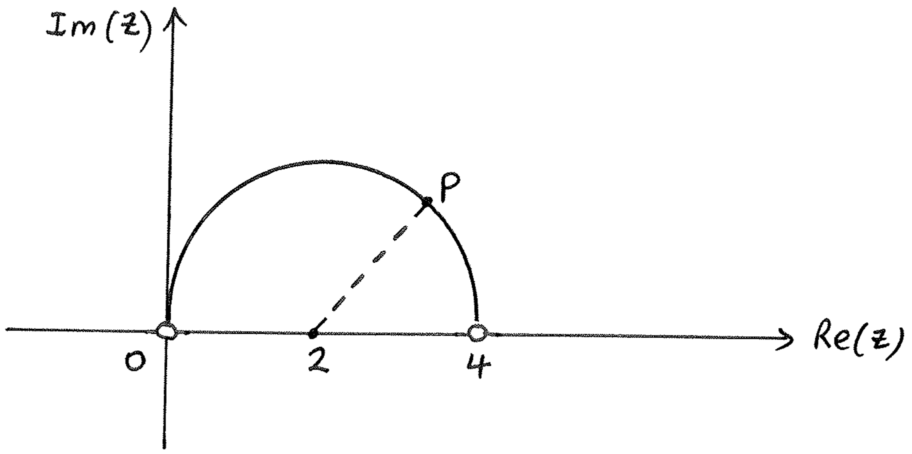
$\therefore$ roots are $z = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}, \ -\frac{1}{4} \pm i\frac{\sqrt{15}}{4}$.

✓

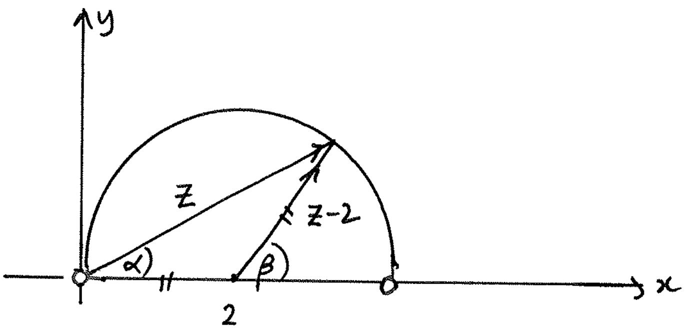
$\boxed{Q12}$ - ctd

(e) $|z-2| = 2$ and $0 < \arg z < \frac{\pi}{2}$

(i)



(ii)



$$\arg(z-2) = k \arg(z^2 - 2z)$$
$$= k \arg z(z-2)$$
$$= k(\arg z + \arg(z-2)$$

Using vectors, we see that

$$\arg(z-2) = 2 \cdot \arg z$$ (exterior < of Δ; equal radii).

Hence $2 \arg z = k(\arg z + 2 \arg z)$

$$2 = 3k$$

$$\therefore \quad k = \frac{2}{3}.$$

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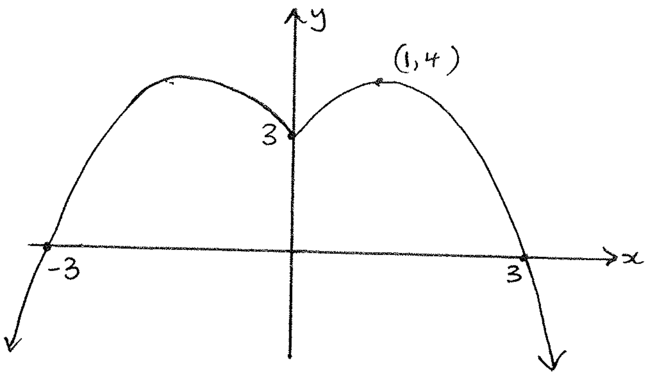
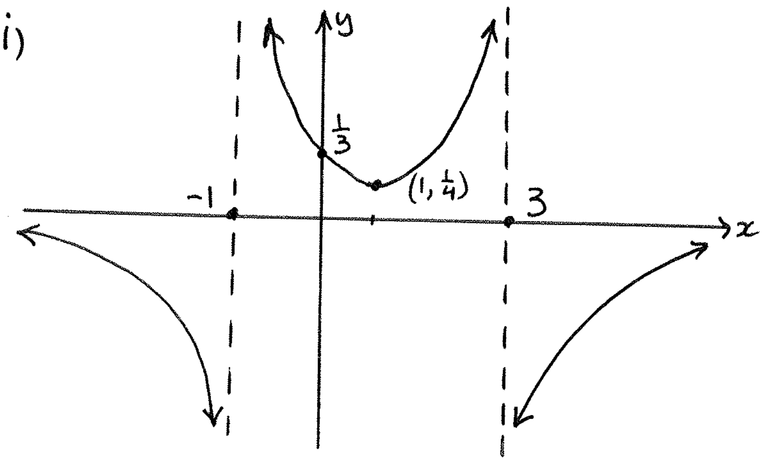
Q13 $f(x) = -(x-3)(x+1)$

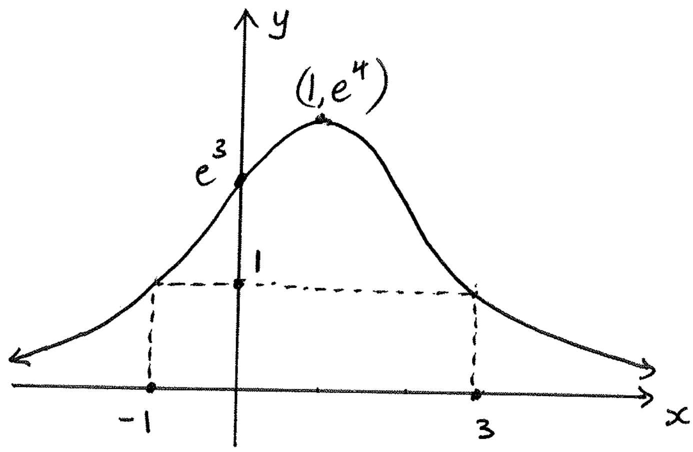
(a) (i)



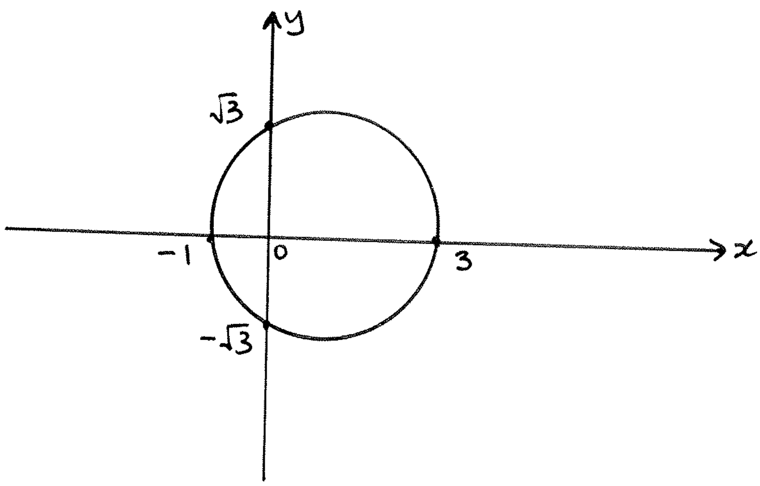
(ii)



(iii)



(iv)



$\boxed{Q13}$ - ctd.

(b) (i) $y = \cos^{-1}(e^x)$

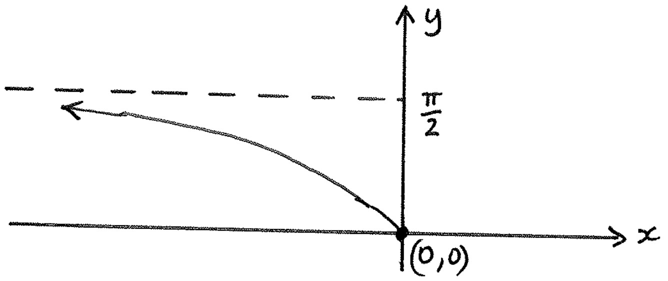
We need $-1 \leq e^x \leq 1 \quad \therefore e^x \leq 1 \quad \therefore x \leq 0$

$\therefore \quad D: \; x \leq 0$

Since $x \leq 0, \quad 0 \leq y < \frac{\pi}{2} \quad$ (as $e^x \to 0^+$ as $x \to -\infty$)

$\therefore \quad R: \; 0 \leq y < \frac{\pi}{2}$

(ii)



(c) $3x^2 + y^2 - 2xy - 8x + 2 = 0$

$\therefore 6x + 2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 8 = 0$

$\therefore \frac{dy}{dx} = \frac{8 - 6x + 2y}{2y - 2x} = \frac{4 - 3x + y}{y - x}$

Thus $\frac{4 - 3x + y}{y - x} = 2$

$\therefore \quad 4 - 3x + y = 2y - 2x \quad \therefore \quad y = 4 - x.$

Sub. in curve equation :-

$3x^2 + 16 - 8x + x^2 - 8x + 2x^2 - 8x + 2 = 0$

$6x^2 - 24x + 18 = 0$

$x^2 - 4x + 3 = 0$

$(x - 3)(x - 1) = 0$

$\therefore x = 3 \rightarrow y = 1$

or $x = 1 \rightarrow y = 3$

$\therefore$ the points are $(3,1)$ and $(1,3)$.

15

Q14

(a) (i) Let $P(x) = (x-\alpha)^n \cdot Q(x)$, $Q(\alpha) \neq 0$.

$\therefore P'(x) = n(x-\alpha)^{n-1} \cdot Q(x) + (x-\alpha)^n \cdot Q'(x)$

$= (x-\alpha)^{n-1} [n \cdot Q(x) + (x-\alpha) \cdot Q'(x)]$

$= (x-\alpha)^{n-1} \cdot Q_1(x)$, where $Q_1(\alpha) \neq 0$

$\therefore \alpha$ is a root of $P'(x)$ of multiplicity $n-1$. ✓

(ii) $P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$ has a triple root

$\therefore P'(x) = 8x^3 + 27x^2 + 12x - 20$ has a double root

$\therefore P''(x) = 24x^2 + 54x + 12$ has a 1-fold root. ✓

$\therefore 24x^2 + 54x + 12 = 0$

$4x^2 + 9x + 2 = 0$

$(4x+1)(x+2) = 0$ $\therefore x = -\frac{1}{4}$ or -2. ✓

Subbing $x = -2$,

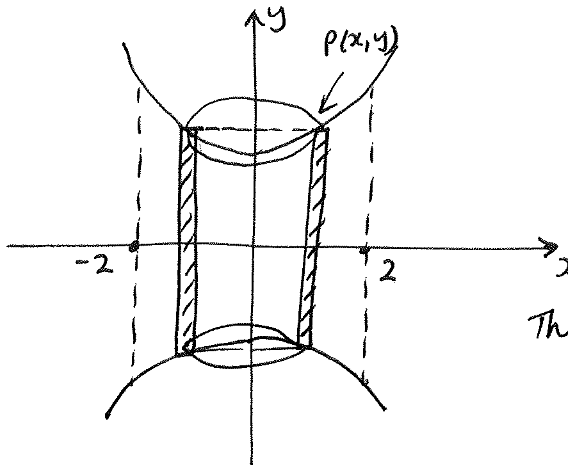
$P(-2) = 2(-2)^4 + 9(-2)^3 + 6(-2)^2 - 20(-2) - 24 = 0$

$\therefore x = -2$ is the triple root.

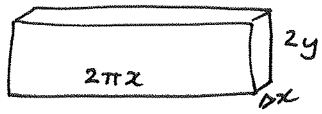
By inspection, $P(x) = (x+2)^3 \cdot (2x-3)$ ✓

$\boxed{Q14}$ - ctd

(b)



A shell at $P(x,y)$ has height $2y$ and curved surface area $2\pi x$.



Thus $\Delta V \doteqdot 2\pi x \cdot 2y \cdot \Delta x$

Now $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1 \quad \therefore y^2 = 9\left(1 + \dfrac{x^2}{4}\right)$

$\therefore y = \pm \frac{3}{2}\sqrt{4+x^2}$

$$\therefore \Delta V = 2\pi x \cdot 3\sqrt{4+x^2}\ \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{x=2} 6\pi x\sqrt{4+x^2}\ \Delta x$$

$$V = 6\pi \int_0^2 x\sqrt{4+x^2}\ dx$$

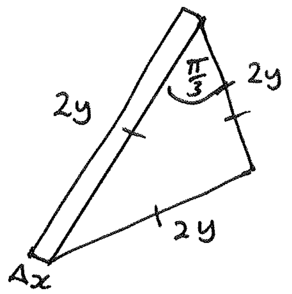
$$= 3\pi \int_0^2 2x\sqrt{4+x^2}\ dx$$

$$= 3\pi \left[\frac{2}{3}(4+x^2)^{\frac{3}{2}}\right]_0^2$$

$$V = 2\pi\left[8^{\frac{3}{2}} - 4^{\frac{3}{2}}\right]$$

$$\therefore V = 16\pi(2\sqrt{2} - 1)\ u^3.$$

$\boxed{Q14}$ - ctd. (c)



Area of each cross-sectional slice is $\frac{1}{2}(2y)^2 \sin\frac{\pi}{3} = \sqrt{3}\,y^2$

$\therefore \Delta V \doteqdot \sqrt{3}\,y^2 \cdot \Delta x$

$\therefore \Delta V \doteqdot \sqrt{3}\,(4-x^2)\,\Delta x$ ✓

Thus $V = \lim_{\Delta x \to 0} \sum_{x=-2}^{+2} \sqrt{3}\,(4-x^2)\,\Delta x$

$$V = \sqrt{3}\int_{-2}^{2}(4-x^2)\,dx = 2\sqrt{3}\int_{0}^{2}(4-x^2)\,dx$$

$$\therefore V = 2\sqrt{3}\left[4x - \frac{x^3}{3}\right]_0^2$$

$$= 2\sqrt{3}\left(8 - \frac{8}{3}\right) \qquad \therefore V = \frac{32\sqrt{3}}{3}\ u^3.$$ ✓✓

(d) $P(x) = x^4 - 2x^3 + 3x^2 - 4x + 1$

(i) $\alpha + \beta + \gamma + \delta = -(-2) = 2$

$(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) = 3$ ✓

So $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= 2^2 - 2(3)$

$= -2$ ✓

(ii) Since $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 0$

$\therefore$ at least one of $\alpha, \beta, \gamma, \delta$ is non-real.

But since the coefficients of $P(x)$ are real, the conjugate of this non-real root is also a root.

What of the remaining 2 roots?

We observe that $P(-1) = 1 + 2 + 3 + 4 + 1 = 11 > 0$

and $P(1) = 1 - 2 + 3 - 4 + 1 = -1 < 0$.

$\therefore$ Since $P(x)$ is continuous, and $P(-1)$ and $P(1)$ have opposite signs, $y = P(x)$ crosses the x axis

$\therefore$ $P(x)$ has a real root. But the remaining root must also be real, else it would have a conjugate that's a root. $\therefore$ exactly 2 real roots ✓✓

15

$\boxed{Q15}$ (a) $P\left(5p, \frac{5}{p}\right)$, $Q\left(5q, \frac{5}{q}\right)$; $p, q > 0$

(i) $$m_{PQ} = \frac{\frac{5}{q} - \frac{5}{p}}{5q - 5p} = -\frac{1}{pq}$$ ✓

$\therefore$ PQ equation is $y - \frac{5}{p} = \frac{-1}{pq}(x - 5p)$

$\therefore \quad pqy - 5q = -x + 5p$

$\therefore \quad x + pqy = 5(p+q)$. ✓

(ii) For tangent at P, let $q \to p$

$\therefore \quad x + p^2 y = 5(2p)$

$\therefore \quad x + p^2 y = 10p$

Likewise, tangent at Q is $x + q^2 y = 10q$. ✓

(iii) For R, solve tangents simultaneously:

$y(p^2 - q^2) = 10(p - q)$

$\therefore$ as $p \neq q$, $\quad y = \frac{10}{p+q}$.

Thus $x = 10p - \frac{10p^2}{p+q} = \frac{10pq}{p+q}$

$\therefore \quad R = \left(\frac{10pq}{p+q}, \frac{10}{p+q}\right)$. ✓✓

(iv) PQ secant is $x + pqy = 5(p+q)$

$\therefore$ if goes thru $(15, 0)$, $15 + 0 = 5(p+q)$

$\therefore \quad p + q = 3$.

Thus $R = \left(\frac{10pq}{3}, \frac{10}{3}\right)$

So locus is $y = \frac{10}{3}$. ✓

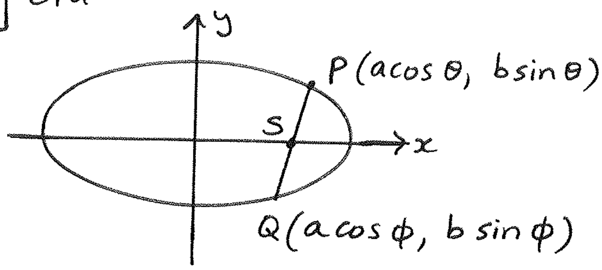
But $x > 0$, since $pq > 0$, and since the intersection cannot occur inside the hyperbola, $\therefore x < 7.5$.

$\therefore$ Locus is $y = \frac{10}{3}$, $0 < x < 7.5$. ✓

[<u>Note</u>: the <u>tangent</u> at $\left(7.5, \frac{10}{3}\right)$ on H goes thru $(15, 0)$]

Q15 ctd.

(b)



$S = (ae, 0)$.

Since PQ is a focal chord, S lies on PQ.

$\therefore \quad m_{PS} = m_{PQ}$

$$\therefore \quad \frac{b\sin\theta - 0}{a\cos\theta - ae} = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}$$ ✓

$$\therefore \quad \frac{\cancel{b}\sin\theta}{\cancel{a}(\cos\theta - e)} = \frac{\cancel{b}(\sin\theta - \sin\phi)}{\cancel{a}(\cos\theta - \cos\phi)}$$

$$\therefore \quad \sin\theta\,\cancel{\cos\theta} - \sin\theta\cos\phi = \cos\theta\,\cancel{\sin\theta} - \cos\theta\sin\phi - e(\sin\theta - \sin\phi)$$

$$\therefore \quad -\sin\theta\cos\phi + \cos\theta\sin\phi = -e(\sin\theta - \sin\phi)$$

$$\therefore \quad \sin\theta\cos\phi - \cos\theta\sin\phi = e(\sin\theta - \sin\phi)$$

$$\therefore \quad \sin(\theta - \phi) = e(\sin\theta - \sin\phi)$$

$$\therefore \quad e = \frac{\sin(\theta - \phi)}{\sin\theta - \sin\phi}$$, as required. ✓✓

[ALT. Equation of PQ is

$$y - b\sin\theta = \frac{b\sin\theta - b\sin\phi}{a\cos\theta - a\cos\phi}(x - a\cos\theta).$$

Sub. in $S(ae, 0)$, and rearrange correctly.]

$\boxed{Q15}$ - ctd.

(c) We have $T_n = 2T_{n-1} - 2T_{n-2}$, $n = 3, 4, 5 \ldots$

and $T_1 = 2$, $T_2 = 0$. To prove: $T_n = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}$.

When $n=1$: $(\sqrt{2})^{1+2} \cos \frac{1(\pi)}{4} = 2\sqrt{2} \cdot \frac{1}{\sqrt{2}} = 2$

$\therefore$ true for $n=1$

When $n=2$: $(\sqrt{2})^{2+2} \cos \frac{2\pi}{4} = 4 \times 0 = 0$

$\therefore$ true for $n=2$ (*)

✓ proves for $n=1,2$

Assume true for $n \le k$

i.e. assume $T_n = (\sqrt{2})^{n+2} \cos \frac{n\pi}{4}$, $n = 1, 2, 3, \ldots k$.

Then look at $n = k+1$:

$T_{k+1} = 2 \cdot T_k - 2 \cdot T_{k-1}$ by the recursive definition.

✓ uses definition

$= 2(\sqrt{2})^{k+2} \cdot \cos \frac{k\pi}{4} - 2(\sqrt{2})^{(k-1)+2} \cos \frac{(k-1)\pi}{4}$

by our assumption.

✓ rewrites in cosines

$$= (\sqrt{2})^{k+3}\left[\sqrt{2} \cos \frac{k\pi}{4} - \cos\left(\frac{k\pi}{4} - \frac{\pi}{4}\right)\right]$$

$$= (\sqrt{2})^{k+3}\left[\frac{2}{\sqrt{2}} \cos \frac{k\pi}{4} - \left(\cos \frac{k\pi}{4} \cos \frac{\pi}{4} + \sin \frac{k\pi}{4} \sin \frac{\pi}{4}\right)\right]$$

$$= (\sqrt{2})^{k+3}\left[\frac{2}{\sqrt{2}} \cos \frac{k\pi}{4} - \frac{1}{\sqrt{2}} \cos \frac{k\pi}{4} - \frac{1}{\sqrt{2}} \sin \frac{k\pi}{4}\right]$$

$$= (\sqrt{2})^{k+3}\left[\frac{1}{\sqrt{2}} \cos \frac{k\pi}{4} - \frac{1}{\sqrt{2}} \sin \frac{k\pi}{4}\right]$$

$$= (\sqrt{2})^{k+3}\left[\cos \frac{\pi}{4} \cos \frac{k\pi}{4} - \sin \frac{\pi}{4} \sin \frac{k\pi}{4}\right]$$

$$= (\sqrt{2})^{k+3}\left[\cos\left(\frac{k\pi}{4} + \frac{\pi}{4}\right)\right]$$

$$\therefore T_{k+1} = (\sqrt{2})^{(k+1)+2} \cos \frac{(k+1)\pi}{4}$$

✓✓

$\therefore$ if true for $n = 1, 2, \ldots k$ it's true for $n = k+1$.
Hence, by induction & (*), true for $n = 1, 2, 3 \ldots$

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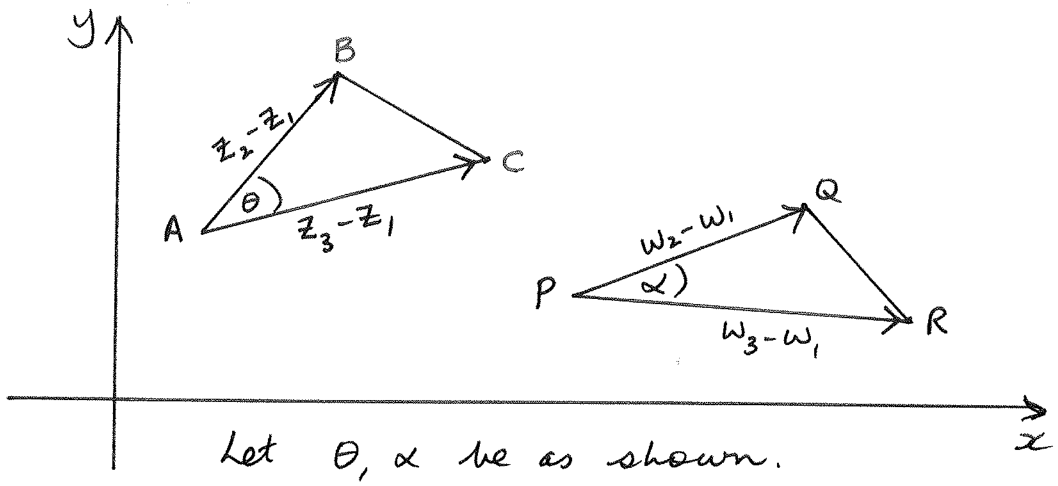
Q16

(a)



Let θ, α be as shown.

Given $$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}$$

$$\therefore \left| \frac{z_2 - z_1}{z_3 - z_1} \right| = \left| \frac{w_2 - w_1}{w_3 - w_1} \right|$$

$$\therefore \frac{|z_2 - z_1|}{|z_3 - z_1|} = \frac{|w_2 - w_1|}{|w_3 - w_1|}$$

$$\therefore \frac{AB}{AC} = \frac{PQ}{PR}.$$

✓ uses modulus

Also, $$\operatorname{Arg}\left(\frac{z_2 - z_1}{z_3 - z_1}\right) = \operatorname{Arg}\left(\frac{w_2 - w_1}{w_3 - w_1}\right)$$

$$\therefore \operatorname{Arg}(z_2 - z_1) - \operatorname{Arg}(z_3 - z_1) = \operatorname{Arg}(w_2 - w_1) - \operatorname{Arg}(w_3 - w_1)$$

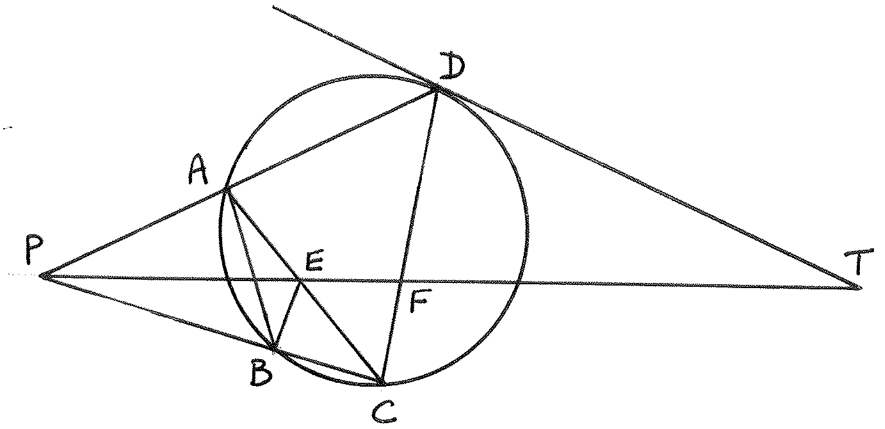
$$\therefore \theta = \alpha$$

✓ uses argument

Thus $\Delta ABC \,|||\, \Delta PQR$, since 2 pairs of corresponding sides are in the same ratio, and the included angle is equal.

✓ correct geometrical reason

Q16 - ctd



(i)

TD = TF (given)

∴ $\angle TFD = \angle TDF$ (base angles of isosceles triangle are equal)

$\angle TDF = \angle CAD$ (angle between tangent and chord at point of contact equals angle in alternate segment)

∴ $\angle TFD = \angle CAD$

∴ AEFD is cyclic (exterior angle equals interior opposite angle) #

√√

(ii) $\angle PEA = \angle ADF$ (exterior angle of cyclic quad AEFD equals interior opposite angle)

$\angle PBA = \angle ADF$ (exterior angle of cyclic quad ABCD equals interior opposite angle)

∴ $\angle PEA = \angle PBA$

√√

∴ PBEA is cyclic (interval AP subtends equal angles at points E & B which are on the same side of AP). #

√

$\boxed{Q16}$ ctd

(c) $I_n = \int_0^1 (1-x^2)^n \, dx$ & $J_n = \int_0^1 x^2(1-x^2)^n \, dx$

(i) $I_n = \int_0^1 (1-x^2)^n \cdot \frac{d}{dx}(x) \, dx$

$= \left[x(1-x^2)^n\right]_0^1 - \int_0^1 -2nx^2(1-x^2)^{n-1} \, dx$ ✓

$= 0 + 2n \int_0^1 x^2(1-x^2)^{n-1} \, dx$

$\therefore I_n = 2n \cdot J_{n-1}$ ✓

(ii) from (i),

$I_n = 2n \int_0^1 x^2(1-x^2)^{n-1} \, dx$

$= -2n \int_0^1 \left[(1-x^2)-1\right] \cdot (1-x^2)^{n-1} \, dx$ ✓

$= -2n \cdot (I_n - I_{n-1})$

$= -2n \cdot I_n + 2n \cdot I_{n-1}$

$\therefore (2n+1) \cdot I_n = 2n \cdot I_{n-1}$

$\therefore I_n = \frac{2n}{2n+1} \cdot I_{n-1}$ ✓

P.T.O.

Q16 ctd.

(c) (iii)

$$J_n = \int_0^1 x^2(1-x^2)^n\,dx$$

$$= -\int_0^1 \left[(1-x^2)-1\right](1-x^2)^n\,dx$$

$$= -\left(I_{n+1} - I_n\right)$$

$$\therefore J_n = I_n - I_{n+1}$$

$$= I_n - \frac{2(n+1)}{2(n+1)+1}\cdot I_n \qquad \text{from (ii)}$$

$$= \frac{(2n+3)\cdot I_n - (2n+2)I_n}{2n+3}$$

$$\therefore J_n = \frac{1}{2n+3}\cdot I_n$$

(iv) using (iii) and (i),

$$J_n = \frac{1}{2n+3}\cdot I_n$$

$$= \frac{1}{2n+3}\cdot 2n\cdot J_{n-1}$$

i.e. $$J_n = \frac{2n}{2n+3}\cdot J_{n-1}$$

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